Short-Circuited Line

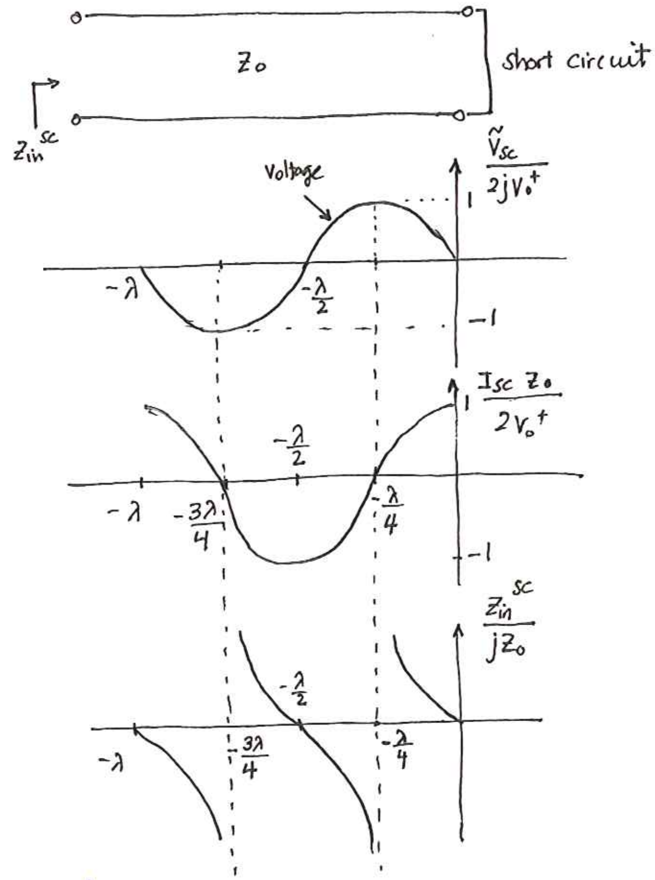
$$\tilde{V}_{sc}(z) = V_0^+\left(e^{-j\beta z} + \overset{\overset{-1}{\uparrow}}{\Gamma} e^{j\beta z}\right)$$

$$= -2jV_0^+ \sin\beta z$$

$$\tilde{I}_{sc}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} + e^{j\beta z}\right) = \frac{2V_0^+}{Z_0}\cos\beta z$$

$$Z_{in}^{sc} = \frac{V_{sc}(-\ell)}{I_{sc}(-\ell)} = \frac{+2jV_0^+\sin\beta\ell}{\frac{2V_0^+}{Z_0}\cos\beta\ell} = jZ_0\tan\beta\ell$$

$$\boxed{Z_{in}^{sc} = jZ_0\tan\beta\ell} \quad (m)$$



$Z_{in} = R_{in} + jX_{in} = jZ_0\tan\beta\ell \rightarrow R_{in} = 0$

$X_{in} = Z_0\tan\beta\ell$

* If $\beta\ell > 0 \rightarrow X_{in} = \omega L_{eq} = Z_0\tan\beta\ell$

$\rightarrow L_{eq} = \frac{Z_0\tan\beta\ell}{\omega} \rightarrow \ell = \frac{1}{\beta}\tan^{-1}\frac{\omega L_{eq}}{Z_0}$ this is the minimum length that results in an equivalent inductance of L_{eq}. Other lengths are $\ell = \frac{n\lambda}{2} + \frac{1}{\beta}\tan^{-1}\frac{\omega L_{eq}}{Z_0}$

* If $\beta\ell < 0 \rightarrow \frac{1}{j\omega C_{eq}} = jZ_0\tan\beta\ell \rightarrow C_{eq} = -\frac{1}{Z_0\omega\tan\beta\ell}$

The minimum length that results in C_{eq} is:

$$Z_0\tan\beta\ell = \frac{1}{\omega C_{eq}} \rightarrow \tan\beta\ell = \frac{1}{Z_0\omega C_{eq}} \rightarrow \beta\ell = \underset{\smile}{\pi} - \tan^{-1}\left(\frac{1}{Z_0\omega C_{eq}}\right)$$

because $\tan\beta\ell \le 0$

$$\rightarrow \ell = \frac{1}{\beta}\left(\pi - \tan^{-1}\left(\frac{1}{Z_0\omega C_{eq}}\right)\right).$$

Of course for all length of $\ell = \frac{n\lambda}{2} + \frac{1}{\beta}\left(\pi - \tan^{-1}\left(\frac{1}{Z_0\omega C_{eq}}\right)\right)$ this is valid.

Open Circuit Line

$$Z_L = \infty \rightarrow \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = 1 \rightarrow \tilde{V}_{oc}(z) = V_0^+\left(e^{-j\beta z} + e^{j\beta z}\right) = 2V_0^+\cos\beta z$$

$$\tilde{I}_{oc}(z) = \frac{V_0^+}{Z_0}\left(e^{-j\beta z} - e^{j\beta z}\right) = -2j\frac{V_0^+}{Z_0}\sin\beta\ell$$

$$Z_{in}^{oc} = \frac{\tilde{V}_{oc}(-\ell)}{\tilde{I}_{oc}(-\ell)} \Rightarrow$$

$$\boxed{Z_{in}^{oc} = -jZ_0\cot\beta\ell}$$

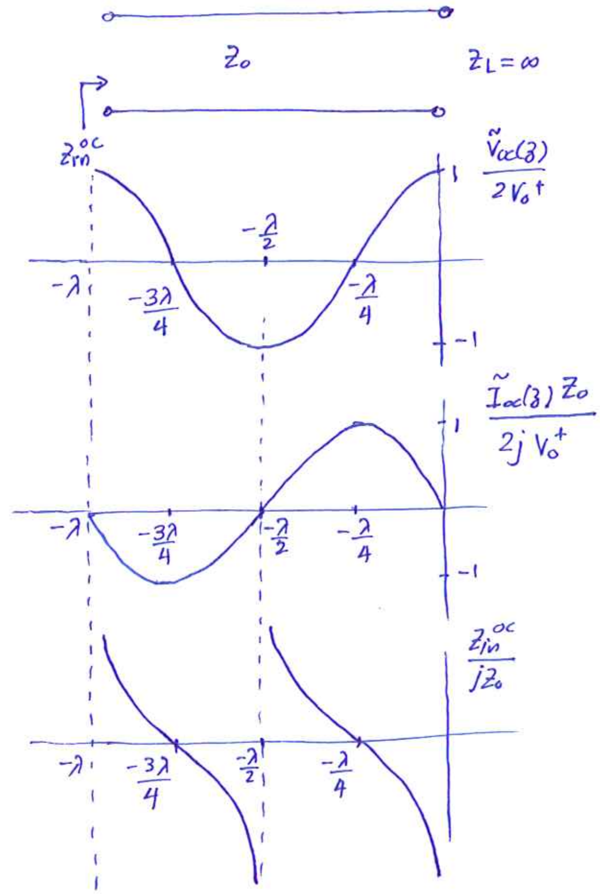
Application of Short-Circuit and Open-circuit Measurement :

Combination of Z_{sc} and Z_{oc} measurement can be used to determine the characteristic impedance of the line Z_0 and its phase constant β:

$$\left.\begin{aligned} Z_{in}^{sc} &= jZ_0 \tan\beta l \\ Z_{in}^{oc} &= -jZ_0 \cot\beta l \end{aligned}\right\} \quad \boxed{\begin{aligned} Z_0 &= \sqrt{Z_{in}^{sc}\, Z_{in}^{oc}} \\ \tan\beta l &= \sqrt{\frac{-Z_{in}^{sc}}{Z_{in}^{oc}}} \end{aligned}}$$

Example : Read example 2-8 page 67.

Open circuited line :

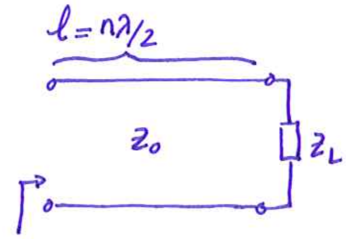


Lines of length $l = \frac{n\lambda}{2}$:

If $l = \frac{n\lambda}{2} \Rightarrow \beta l = \frac{2\pi}{\lambda}\frac{n\lambda}{2} = n\pi \rightarrow \tan\beta l = \tan n\pi = 0$

$$\rightarrow \quad Z_{in} = Z_0 \frac{Z_L + jZ_0 \cancel{\tan\beta l}}{Z_0 + jZ_L \cancel{\tan\beta l}} = Z_L \quad \rightarrow \quad \boxed{Z_{in} = Z_L \quad \text{for } l = n\frac{\lambda}{2}}$$

In other words, half-wavelength line doesn't modify the load impedance.



$Z_{in} = Z_L$

Quarter-wave Transformer :

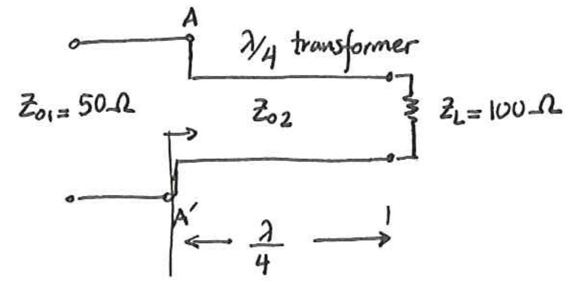
Now take $l = \frac{\lambda}{4} + \frac{n\lambda}{2} \rightarrow \tan\beta l = \tan\left(\frac{2\pi}{\lambda}\frac{\lambda}{4} + \frac{2\pi}{\lambda}\frac{n\lambda}{2}\right) = \tan\left(\frac{\pi}{2} + n\pi\right) = \infty$

$$\rightarrow \quad Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} = Z_0 \frac{jZ_0}{jZ_L} = \frac{Z_0^2}{Z_L}$$

$$\boxed{Z_{in} = \frac{Z_0^2}{Z_L} \quad \text{for } l = \frac{\lambda}{4} + \frac{n\lambda}{2}}$$

Example

we want to match a load impedace of $Z_L = 100\,\Omega$ to a lossless transmission line with $Z_0 = 50\,\Omega$ via a quarter_wave section as shown in the picture. Find the characteristic impedance of the quarter-wave transformer.



$$Z_{in} = \frac{Z_{02}^2}{Z_L} \rightarrow Z_{02} = \sqrt{Z_{in} Z_L} = \sqrt{50 \times 100} = 70.7\,\Omega$$

Matched Transmission Line $Z_L = Z_0$

for a matched line: 1) $Z_L = Z_0 \rightarrow Z_{in} = Z_0$

2) reflectance is zero $\Gamma = 0$

3) all incident power is delivered to the load regardless of l.

Power flow on a lossless transmission line

Instantaneous power: the power carried by the incident wave arrived at the load is equal to:

$$P^i(t) = v^i(t)\, i^i(t)$$

$$= Re\left[\tilde{V}^i e^{j\omega t}\right] \cdot Re\left[\tilde{I}^i e^{j\omega t}\right]$$

$$= Re\left[|V_0^+| e^{j\phi^+} e^{j\omega t}\right] \cdot Re\left[\frac{|V_0^+|}{Z_0} e^{j\phi^+} e^{j\omega t}\right]$$

$$= |V_0^+| \cos(\omega t + \phi^+) \cdot \frac{|V_0^+|}{Z_0} \cos(\omega t + \phi^+)$$

$$= \frac{|V_0^+|^2}{Z_0} \cos^2(\omega t + \phi^+)$$ instantaneous power incident on the load

Similarly using $\Gamma = |\Gamma| e^{j\theta_r}$ we can write down the instantaneous power reflected from the load:

$$P^r(t) = v^r(t) \cdot i^r(t) = -|\Gamma|^2 \frac{|V_0^+|^2}{Z_0} \cos^2(\omega t + \phi^+ + \theta_r)$$

Time Average power

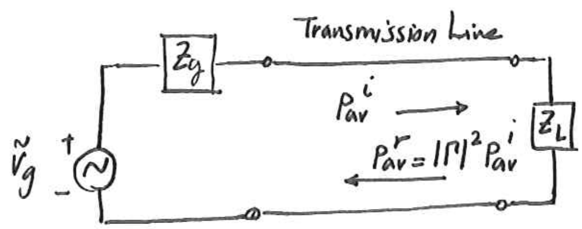
We usually want to know the average power flow in the line than the instantaneous power. To calculate the average power P_{av} we can use phasor or the time-domane form. Let's do both:

Time-domain approach:

$$P^i_{av} = \frac{1}{T}\int_0^T P^i(t)\,dt = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} P^i(t)\,dt = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{|V_0^+|^2}{Z_0}\cos^2(\omega t + \phi^+)\,dt$$

$$\boxed{P^i_{av} = \frac{|V_0^+|^2}{2Z_0}}$$

Similarly for the reflected power: $\boxed{P^r_{av} = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0} = -|\Gamma|^2 P^i_{av}}$



The net power delivered to the load is:

$$\boxed{P_{av} = P^i_{av} + P^r_{av} = \frac{|V_0^+|^2}{2Z_0}\left[1 - |\Gamma|^2\right]}$$

Phasor-Domain Approach:

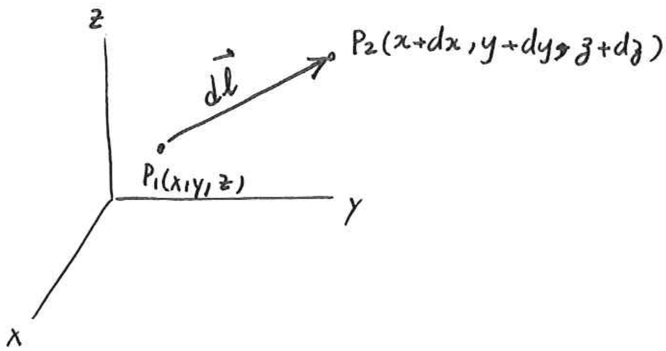
A useful formula for average power in phasor domain is:

$$\boxed{P_{av} = \frac{1}{2} \mathrm{Re}\left[\tilde{V} \cdot \tilde{I}^*\right]}$$

$$P^i_{av} = \frac{1}{2} \mathrm{Re}\left[V_0^+ \cdot \frac{V_0^{+*}}{Z_0}\right] = \frac{|V_0^+|^2}{2Z_0}$$

$$P^r_{av} = \frac{1}{2} \mathrm{Re}\left[\Gamma V_0^+ \cdot \left(\frac{-\Gamma^* V_0^{+*}}{Z_0}\right)\right] = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}$$

Gradient of a Scalar Field:



$$\boxed{\vec{\nabla} T = \text{grad}\, T \triangleq \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}}$$

$$dT = \vec{\nabla} T \cdot \vec{dl}$$

del or gradient operator: $\boxed{\nabla \triangleq \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$ (cartesian)

$$\vec{dl} = \hat{a}_l\, dl \rightarrow \boxed{\frac{dT}{dl} = \nabla T \cdot \hat{a}_l}$$

In Cylindrical and Spherical Coordinates:

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z} \quad \text{(cylindrical)}$$

$$\nabla = \hat{R}\frac{\partial}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi} \quad \text{(spherical)}$$

Divergence of a vector Field:

$$\boxed{\vec{\nabla} \cdot \vec{E} \triangleq \text{div}\, E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}$$

For cylindrical & spherical, refer to the back cover of the book.

Divergence Theorem:

$$\boxed{\int_v \nabla \cdot E\, dv = \oint_S E \cdot ds}$$

Example

Calculate the divergence of each of the following vector fields:

a) $\vec{E} = \hat{x}\, 3x^2 + \hat{y}\, 2z + \hat{z}\, x^2 z$

b) $\vec{E} = \hat{R}\, a^3 \dfrac{\cos\theta}{R^2} - \hat{\theta}\left(a^3 \dfrac{\sin\theta}{R^2}\right)$

Solution: (a) $\vec{\nabla}\cdot\vec{E} = \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z} = 6x + 0 + x^2 = 6x + x^2$

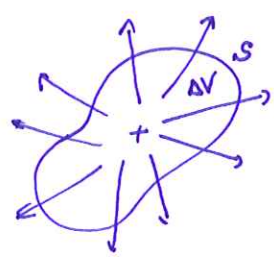
(b) $$\vec{\nabla}\cdot\vec{E} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(E_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial \phi}$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(a^3\cos\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{a^3\sin\theta}{R^2}\right)$$

$$= 0 - \frac{2a^3\cos\theta}{R^3} = -\frac{2a^3\cos\theta}{R^3}$$

Note:

Consider a closed surface in an electric field E:



- $\vec{\nabla}\cdot\vec{E}$ is positive if the net flux out of surface S is positive. In other word, if there is a "source" inside the volume encountered by surface S.

- $\vec{\nabla}\cdot\vec{E}$ is negative if the net flux is inward. In other word if there is a "sink" inside the volume defined by surface S.

- for a uniform field, the same amount of flux enters volume ΔV as it leaves the volume. So $\vec{\nabla}\cdot\vec{E} = 0$

We call E here a "divergenceless" field.